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Toda systems in closed string tachyon condensation

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Abstract: We consider tt^* equations appearing in the study of localized tachyon condensations. They are described by various Toda system when we consider the condensation by the lowest tachyon corresponding to the monomial xy. The tachyon potential is calculated as a solution to these equations. The Toda system appearing in the deformation of $\mathbb{C}^2/\mathbb{Z}_n$ by xy is identical to that of D_n singularity deformed by x. For $\mathbb{C}^3/\mathbb{Z}_n$ with xyz deformation, we find only generic non-simple form, similar to the case appearing in $\mathbb{C}/\mathbb{Z}_5 \to \mathbb{C}/\mathbb{Z}_3$ and we discuss the difficulties in these cases.

Keywords: Tachyon Condensation, Integrable Hierarchies.

Contents

1. Introduction

The study of localized tachyon condensation [1 − 5] has been considered with many interesting developments. The basic picture is that tachyon condensation induces cascade of decays of the orbifolds to less singular ones until the spacetime supersymmetry is restored. Therefore the localized tachyon condensation has a geometric description as the resolution of the spacetime singularities.

Following the line of Vafa's reformulation of the problem in terms of Mirror Landau-Ginzburg theory, we worked out the detailed analysis on the fate of spectrum and the background geometry under the tachyon condensation as well as the question of what is the analogue of c-theorem with the GSO-projection in a series of papers [7 − 9].

The study of tachyon dynamics would be greatly improved if we know the potential that governs the condensation process. In this direction, Dabholkar and Vafa [4] proposed that the tachyon potential is given by the maximal charge and worked out the case describing the decay of $\mathbb{C}^1/\mathbb{Z}_3$ to $\mathbb{C}$. In the previous paper [10], we generalized the result to $\mathbb{C}^2/\mathbb{Z}_n$ for $n = 3, 4, 5$. We encountered solvable non-linear equations called Painleve III as well as its degenerate case. So it is interesting to ask which integrable system is waiting us for general n's, if they are still integrable at all.

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In this paper, we consider $\mathbb{C}^2/\mathbb{Z}_n$ as well as to $\mathbb{C}^1/\mathbb{Z}_n$ for any n and show that the resulting systems are various Toda systems when we consider the condensation by the lowest tachyon corresponding to the monomial xy. We find that the potential is calculated as a solution to Toda equations. One interesting notice is that the Toda system appearing as tt^* equation in the deformation of $\mathbb{C}^2/Z_n$ by xy is identical to that of D_n singularity by x, whose mirror dual geometry is not clear.

The outline of the paper is as follows. In section 2, we summarize the basic concepts to introduce languages. In section 3, we consider the tt^* equations for $\mathbb{C}^1/\mathbb{Z}_n$. In section 4, we calculate the tt^* equations for $\mathbb{C}^2/\mathbb{Z}_n$. In section 5, we discuss and conclude. In appendix A, we calculate for $\mathbb{C}^3/\mathbb{Z}_n \to \mathbb{C}^3$ as well as for $\mathbb{C}\mathbb{Z}_5 \to \mathbb{C}/\mathbb{Z}_3$ and discuss the difficulties in analyzing the resulting equations quantitatively. In appendix B, we give a sample residue calculation for topological metric that is an important ingredient of tt^* equations.

2. Landau-Ginzburg description of orbifold geometry and tt^* equation

The Mirror of $\mathbb{C}^2/\mathbb{Z}_n$ is an orbifolded Landau-Ginzburg (LG) model with potential $W = x^n + y^n$ [12, 13]. The main steps for the tachyon potential is to calculate the tt^* equation and their solutions. In this section we summarize the basic concepts on these material following [2] and [14].

Landau-Ginzburg description of $\mathbb{C}^2/\mathbb{Z}_n$: Here we give a brief summary of Landau-Ginzburg formulation of localized tachyon condensation [2]. For simplicity we take $\mathbb{C}^2/\mathbb{Z}_n$. The orbifold $\mathbb{C}^2/\mathbb{Z}_n$ is defined by the $\mathbb{Z}_n$ action given by equivalence relation

$$(X_1, X_2) \sim (\omega^{k_1} X_1, \omega^{k_2} X_2)\,, \qquad \omega = e^{2\pi i/n}\,. \tag{2.1}$$

We call (k_1, k_2) as the generator of the $\mathbb{Z}_n$ action. The orbifold can be imbedded into the gauged linear sigma model(GLSM) [13]. The vacuum manifold of the latter is described by the D-term constraints

$$-n|X_0|^2 + k_1|X_1|^2 + k_2|X_2|^2 = t\,. \tag{2.2}$$

Its $t \to -\infty$ limit corresponds to the orbifold and the $t \to \infty$ limit is the $O(-n)$ bundle over the weighted projected space WP_{k_1,k_2}. In the latter case, X_0 direction corresponds to the non-compact fiber of this bundle and t plays role of size of the WP_{k_1,k_2}.

By dualizing this GLSM, we get a LG model with a superpotential [12]

$$W = \exp(-Y_1) + \exp(-Y_2) + \exp(-Y_0)\,, \tag{2.3}$$

where twisted chiral fields Y_i are periodic $Y_i \sim Y_i + 2\pi i$ and related to X_i by $Re[Y_i] = |X_i|^2$. Introducing the variable $u_i := e^{-Y_i/n}$, the periodicity of Y_i imposes the identification: $u_i \sim e^{2\pi i/n} u_i$ which necessitate modding out each u_i by $\mathbb{Z}_n$. The D-term constraint is expressed as $e^{-Y_0} = e^{t/n} u_1^{k_1} u_2^{k_2}$ whose invariance requires that only one $\mathbb{Z}_n$ can be independent. The result is usually described by

$$[W = u_1^n + u_2^n + e^{t/n} u_1^{k_1} u_2^{k_2}]//\mathbb{Z}_n\,. \tag{2.4}$$

which describe the mirror Landau-Ginzburg model of the linear sigma model. As $t \to -\infty$ limit, mirror of the orbifold is

$$[W = u_1^n + u_2^n]//\mathbb{Z}_n \,. \tag{2.5}$$

From now on we use x, y instead of u_1, u_2 for notational convenience and fix $(k_1, k_2) = (1, k)$ since it is allowed [8]. Since it is not ordinary Landau-Ginzburg theory but an orbifolded version, the chiral ring structure of the theory is slightly different from that of LG model. For example, the chiral ring of LG model is

$$\{x^i y^j | i, j = 1, 2, \ldots, n-1\} \,, \tag{2.6}$$

while that of orbifolded LG model is

$$\{x^i y^j | (i, j) = l(1, k), l = 1, 2, \ldots, n-1\} \,, \tag{2.7}$$

where $(1, k)$ is the generator of the chiral ring and k defines the orbifold action in $\mathbb{C}^2$:

$$X_1 \to \omega X_1 \,, \qquad X_2 \to \omega^k X_2 \,. \tag{2.8}$$

The former has $(n-1)^2$ elements while the latter has the $n-1$ elements. We confine ourselves to the case where we deform the theory by the lowest charge xy.

tt^* **equations:** here we briefly summarize the content of tt^* equation following [14]. In an $N = 2$ SUSY theory, there are two supersymmetry charges, $\bar{Q}$ and Q. Their property is

$$(\bar{Q})^2 = (Q)^2 = 0 \,, \qquad \{\bar{Q}, Q\} = H \,, \qquad (\bar{Q})^\dagger = Q \,, \tag{2.9}$$

where H is the hamiltonian. Topological theory is obtained by declaring $\bar{Q}$ to be a BRST operator and by identifying the BRST cohomology of $\bar{Q}$ with physical Hilbert space. The ground states $|i\rangle$ satisfy $\bar{Q}|i\rangle = 0$, where we take the space to be a circle with periodic boundary condition. The topological operators ϕ_i are defined to be operators which commute with $\bar{Q}$, i.e. $[\bar{Q}, \phi_i] = 0$, and are called chiral fields. These chiral fields form a ring because of OPE of two of them is $\bar{Q}$-closed.

$$\phi_i \phi_j = C_{ij}^k \phi_k + [\bar{Q}, \Lambda] \,. \tag{2.10}$$

The CPT conjugate operators $\bar{\phi}_i$ commute with Q and are called anti-chiral. There is an one-to-one correspondence between the ground states and the chiral operators: $\phi_i|0\rangle = |i\rangle + \bar{Q}|\chi\rangle$. By definitions, the resulting state is a topological state. By acting the anti-chiral operators we get anti-topological state. The topological metric is defined by the two point function of chiral operators:$\eta_{ij} = \langle j|i\rangle$, and the ground state metric is defined as an inner product of a topological and an anti-topological state: $\langle \bar{j}|i\rangle = g_{i\bar{j}}$. They satisfy the reality condition:

$$\eta^{-1} g (\eta^{-1})^* = I \,. \tag{2.11}$$

The two point function η_{ij} can be calculated by the Grothendieck residue

$$\begin{aligned} \eta_{ij} = \langle \phi_i \phi_j \rangle &= Res[\phi_i \phi_j], \\ Res[\phi] &= \frac{1}{(2\pi i)^n} \int_\Gamma \frac{\phi(X) dX^1 \wedge \cdots \wedge dX^n}{\partial_1 W \partial_2 W \cdots \partial_n W} \,, \end{aligned} \tag{2.12}$$

where W is the superpotential.

Now let us consider the perturbation of the action

$$S = \int d^2zd^4\theta K + \left(\int d^2zd^2\theta W + \text{c.c.}\right) \tag{2.13}$$

by the chiral operators with

$$\delta S = \delta t_i \int d^2zd^2\theta\phi_i + \text{c.c.}, \tag{2.14}$$

where t_i correspond to the (complex) couplings in the theory. As t_i change the Ramond vacua change. The method of the computation of the metric $g_{i\bar{j}}$ is to study its behavior under perturbatuions which preserves the $N = 2$ SUSY.

Analogous to Berry connection let us introduce the connections A_i on the coupling constant space (holomorphic parameter space) given by

$$A_{ia\bar{b}} = \langle\bar{b}|\partial_i|a\rangle\,. \tag{2.15}$$

This connection measures the way the ground state subsector varies in the Hilbert space as the couplings change. Under a coupling constant dependent change of basis for the vacua, A_i transforms as a gauge field.

If we define the covariant derivative by $D_i = \partial_i - A_i$, then the metric $g_{i\bar{j}}$ is covariantly constant $D_i g_{i\bar{j}} = 0 = \bar{D}_i g_{i\bar{j}}$. One can prove the following equations [14]

$$\begin{aligned} [D_i, D_j] = [\bar{D}_i, \bar{D}_j] &= 0\,, \\ [D_i, \bar{D}_j] &= -[C_i, \bar{C}_j]\,, \end{aligned} \tag{2.16}$$

which are called tt^* equations. If we choose holomorphic gauge $A_{\bar{i}} = 0$ the second equation can be written as

$$\bar{\partial}_i(g\partial_j g^{-1}) - [C_j, g(C_i)^\dagger g^{-1}] = 0\,, \tag{2.17}$$

which will be our main concern here.

3. tt^* equations for $\mathbb{C}^1/\mathbb{Z}_n$

The tt^* equations for the one field LG model under the deformation of most relevant operators (x, x^2) were calculated by Cecotti and Vafa [14] and the results were various Toda systems. One can adopt their result to discuss the localized tachyon condensation in $\mathbb{C}^1/\mathbb{Z}_n$. The only thing one has to be careful is the renaming $n + 1 \to n$ and the shift in power in the chiral ring elements associated with shift $\partial_x \to x\partial_x$. As a consequence, the evenness and oddness of various quantities are all reversed. So it is useful to rewrite the tt^* equations in the present context and present notations.

3.1 Deformation by x: condensation of the most tachyonic state

First we discuss the tachyon condensation by the most tachyonic state which has mass $m^2 = 2\alpha'(1/n - 1)$. The monomial representation of the corresponding operator is simply given by x whose NS-charge is $1/n$. So the tachyon condensation in the LG description is described by the superpotenial

$$W = \frac{x^n}{n} - tx\,. \tag{3.1}$$

Since the basic variable is $\log x$, the chiral ring is

$$\mathcal{R} = \mathbb{C}[x]/\partial_{\log x} W = \{x, x^2, \ldots, x^{n-1}\}\,. \tag{3.2}$$

The non-vanishing components of the topological metric η are

$$\eta_{i,n-i} = 1\,, \qquad \text{for } i = 1, 2, \ldots, n-1\,. \tag{3.3}$$

Let $a_i := g_{i\bar{i}} = \langle \bar{i} | i \rangle$. The reality condition gives

$$a_i a_{n-i} = 1\,, \qquad \text{if } n = 2m+1\,, \quad a_m = 1\,. \tag{3.4}$$

The non-vanishing components of structure constant matrix C_t are

$$(C_t)_1^2 = (C_t)_2^3 = \cdots = (C_t)_{n-2}^{n-1} = 1\,, \qquad (C_t)_{n-1}^1 = t\,. \tag{3.5}$$

With these, the tt^* equations can be calculated.[1]

$$\begin{aligned}
-\partial_{\bar{t}}\partial_t \log a_1 &= \frac{a_2}{a_1} - |t|^2 \frac{a_1}{a_{n-1}} \\
-\partial_{\bar{t}}\partial_t \log a_i &= \frac{a_{i+1}}{a_i} - \frac{a_i}{a_{i-1}}\,, \qquad i = 2, \ldots, n-2 \\
-\partial_{\bar{t}}\partial_t \log a_{n-1} &= |t|^2 \frac{a_1}{a_{n-1}} - \frac{a_{n-1}}{a_{n-2}}\,.
\end{aligned} \tag{3.6}$$

By introducing new variables

$$\begin{aligned}
q_i &= \log a_i - \frac{2i-n}{2(n-1)} \log |t|^2\,, \\
z &= \frac{n-1}{n} t^{n/(n-1)}\,,
\end{aligned} \tag{3.7}$$

and $q_n = q_1$, $q_0 = q_{n-1}$, the equations given in eqs. (3.6) take the form

$$\partial_z \partial_{\bar{z}} q_i + e^{q_{i+1} - q_i} - e^{q_i - q_{i-1}} = 0\,, \qquad \text{for } i = 1, \ldots, n-1\,. \tag{3.8}$$

which is the A_{n-2} Toda equations. By imposing the reality condition $q_i + q_{n-i} = 0$, we can reduce the system to C_m (if $n = 2m+1$) or BC_m (if $n = 2m+2$) Toda system.

[1] For $\mathbb{C}^1/\mathbb{Z}_n$ problem, some of the tt^* equations found by Cecotti and Vafa in [14] can be adopted for our purpose, while for $\mathbb{C}^2/\mathbb{Z}_n$ we really have to workout the corresponding systems.

3.2 Deformation by x^2

$$W = \frac{x^n}{n} - \frac{t}{2}x^2 \,. \tag{3.9}$$

The chiral ring is the same as before. The topological metric η are

$$\eta_{i,j} = \delta_{i+j,n} + t\delta_{i,n-1}\delta_{j,n-1} \,, \qquad \text{for } i,j = 1,2,\ldots,n-1 \,. \tag{3.10}$$

The non-vanishing components of the ground state metric g_{ij} is

$$g_{i,\bar{i}} = a_i \,, \qquad g_{n-1,\bar{1}} = b \,, \qquad g_{1,\overline{n-1}} = \bar{b} \,. \tag{3.11}$$

The reality condition gives

$$\begin{aligned} a_i a_{n-i} &= 1 \,, \qquad \text{for } i = 2,\ldots,n-2 \\ a_{n-1} &= \frac{1}{a_1} + \frac{|t|^2}{4}a_1 \,, \\ b &= \frac{1}{2}ta_1 \\ a_m &= 1 \qquad \text{if } n = 2m+1 \,, \end{aligned} \tag{3.12}$$

The non-vanishing components of structure constant matrix C_t are

$$(C_t)_1^3 = (C_t)_2^4 = \cdots = (C_t)_{n-3}^{n-1} = 1 \,, \qquad (C_t)_{n-2}^2 = (C_t)_{n-1}^3 = t \,. \tag{3.13}$$

According to whether n is even or odd, the tt^* equations are of different type.

For $n = 2m+1$, we define

$$\begin{aligned} q_i &= -\log a_{2i-1} \qquad \text{for } i = 1,2,\ldots,\left[\frac{(m+1)}{2}\right] \,, \\ &= \log a_{2(m-i)+2} \qquad \text{for } i = \left[\frac{(m+1)}{2}\right] + 1,\ldots,m \end{aligned} \tag{3.14}$$

Then the equations are

$$\begin{aligned} \partial\bar{\partial}q_1 &= e^{q_1-q_2} - \frac{1}{4}|t|^2 e^{-(q_1+q_2)} \,, \\ \partial\bar{\partial}q_2 &= e^{q_2-q_3} - e^{q_1-q_2} - \frac{1}{4}|t|^2 e^{-(q_1+q_2)} \,, \\ \partial\bar{\partial}q_i &= e^{q_i-q_{i+1}} - e^{q_{i-1}-q_i} \,, \\ \partial\bar{\partial}q_m &= |t|^2 e^{2q_m} - e^{q_{m-1}-q_m} \,. \end{aligned} \tag{3.15}$$

It is easy to re-scale the variables to make the equation as a standard $\tilde{B}_m := D^T(\mathrm{SO}(2m+1))$ Toda form.

For even n, the tt^* equations decouples into two independent Toda systems and we have two cases according to whether $n/2$ is even or odd.

1. $n = 4m$: $a_1, a_3, \ldots, a_{2m-1}$ satisfies $\tilde{B}_m$ Toda equations, while $a_2, a_4, \ldots, a_{2m-2}$ satisfies BC_{m-1} Toda.

2. $n = 4m+2$: $a_1, a_3, \ldots, a_{2m-1}$ satisfies B_m Toda equations, while $a_2, a_4, \ldots, a_{2m}$ satisfies C_m Toda.

4. tt^* equations for $\mathbb{C}^2/\mathbb{Z}_n$

Now we consider $\mathbb{C}^2/\mathbb{Z}_{n(1,1)}$ with the generator xy and we consider the condensation of most tachyonic operator, namely the generator xy itself. Then the superpotential is given by

$$W = \frac{x^n}{n} + \frac{y^n}{n} - txy\,. \tag{4.1}$$

Since the tt^* equations for this case has never been calculated, we do it here. Before we consider the mirror of $\mathbb{C}^2/\mathbb{Z}_n$, namely the orbifolded LG model, we first consider the generality of LG model itself without the orbifold action.

4.1 General aspects of LG model without orbifold action

To see what is the non-vanishing elements of the ground state metric $g_{i\bar{j}}$ and the topological metric η_{ij}, we first consider the discrete symmetries of the superpotential.

Under the transformation

$$x \to \omega^a x \qquad \text{and} \qquad y \to \omega^b y \tag{4.2}$$

the superpotential W is symmetric upto an over all phase ω^* which is cancelled by transforming the $\theta \to \omega^{(a+b)/2}\theta$. By requiring invariance of the action we have

$$\omega^{an} = \omega^{bn} = \omega^{a+b}\,. \tag{4.3}$$

Without loss of generality, we can set:

$$a = 1\,, \qquad b = n-1\,, \qquad \omega^{n(n-2)} = 1\,, \tag{4.4}$$

up to an equivalence class. Now η_{ij} is given by the Griffith residue

$$\eta_{ij} = \langle x^{i_1}y^{j_1} \cdot x^{i_2}y^{j_2}\rangle = \int \frac{dx}{x} \wedge \frac{dy}{y} \frac{x^{i_1+i_2}y^{j_1+j_2}}{(x^n - txy)(y^n - txy)}\,. \tag{4.5}$$

The non-vanishing components should be invariant under the above symmetry transformations. This requires

$$(i_1 + i_2 - 2) + (n-1)(j_1 + j_2 - 2) \equiv 0 \bmod n(n-2)\,. \tag{4.6}$$

The selection rule is given by the solutions to this equation:

$$i_1 + i_2 = j_1 + j_2 = I \equiv 2 \bmod (n-2)\,. \tag{4.7}$$

For the actual value, we perform explicit residue calculations, whose example is given in the appendix. We found that

- $I = 2$ case: We have one solution $i_1 = i_2 = 1, j_1 = j_2 = 1$. However, it turns out that $\eta = 0$ in this case.
- $I = n$ case: $i_1 = j_1 = i, i_2 = j_2 = n-1-i, i = 1, \ldots, n-1$. These are $n-1$ cross diagonal elements. All of them are 1.

- $I = 2(n-1)$ case: There is only one element $i_1 = j_1 = i_2 = j_2 = n-1$, and the value is $\eta_{ij} = t^2$.

One can find similar selection rule for the metric element $g_{i\bar{j}} = \langle x^{i_1} y^{j_1} x^{\bar{i}_2} y^{\bar{j}_2} \rangle$. For non-vanishing elements we need

$$\omega^{i_1 - i_2 + (n-1)j_1 - (n-1)j_2} = 1 \qquad \text{with } \omega^{n(n-2)} = 1\,, \tag{4.8}$$

which gives

$$i_1 - i_2 + (n-1)(j_1 - j_2) \equiv 0 \bmod n(n-2)\,. \tag{4.9}$$

As consequences we have

$$i_1 - i_2 = j_1 - j_2 \quad \equiv 0 \bmod (n-2)\,. \tag{4.10}$$

There are two cases:

- $i_1 = i_2,\ j_1 = j_2$. These are the diagonal elements of $g_{i\bar{j}}$. We define $a_{ij} := \langle \bar{x}^i \bar{y}^j | x^i y^j \rangle$.
- $j_1 - j_2 = i_1 - i_2 = \pm(n-2)$, namely

$$i_1 = j_1 = n-1 \qquad i_2 = j_2 = 1\,, \tag{4.11}$$

or

$$i_1 = j_1 = 1\,, \qquad i_2 = j_2 = n-1\,, \tag{4.12}$$

which are the two corner-most off-diagonal elements. we call it as $\bar{b}$ and b respectively.

The calculation of tt^* equation for general LG model is cumbersome. We now simplify our life by performing the orbifold action. The simplification is given by reducing the number of chiral ring elements from $(n-1)^2$ to $n-1$.

4.2 Orbifolded LG: the mirror of $\mathbb{C}^2/\mathbb{Z}_n$

The chiral ring in this case is generated by xy:

$$\mathcal{R} = \{xy, (xy)^2, \ldots, (xy)^{n-1}\}\,. \tag{4.13}$$

Let

$$a_i = \langle (\bar{x}\bar{y})^i | (xy)^i \rangle = e^{-q_i}\,. \tag{4.14}$$

The topological metric can be calculated to give

$$\eta = \eta_{ij} = \delta_{i+j,n} + t^2 \delta_{i,n-1} \delta_{j,n-1}\,. \tag{4.15}$$

The structure constants are

$$(C_t)_i^j = \delta_{i,j-1} + t^2 \delta_{i,n-1} \delta_{j,2}\,. \tag{4.16}$$

The reality condition gives

$$a_i a_{n-i} = 1 \quad (\text{for } i = 2, \ldots, n-1)\,, \qquad a_{n-1} = \frac{1}{a_1} + \frac{|t|^4 a_1}{4}\,. \tag{4.17}$$

With all these, we can calculate the tt^* equations. Let's introduce a parameter s by

$$n = 2m + 2 - s\,, \qquad s = 0, 1\,. \tag{4.18}$$

Then tt^* equation can be written as

$$\begin{aligned}
-\partial_{\bar{t}}\partial_t \log a_1 &= \frac{a_2}{a_1} - \frac{1}{4}|t|^4 a_1 a_2 \\
-\partial_{\bar{t}}\partial_t \log a_2 &= \frac{a_3}{a_2} - \frac{a_2}{a_1} - \frac{1}{4}|t|^4 a_1 a_2 \\
-\partial_{\bar{t}}\partial_t \log a_i &= \frac{a_{i+1}}{a_i} - \frac{a_i}{a_{i-1}}\,, \qquad i = 3, \ldots, m-1 \\
-\partial_{\bar{t}}\partial_t \log a_m &= a_m^{-(1+s)} - \frac{a_m}{a_{m-1}}\,.
\end{aligned} \tag{4.19}$$

Let's re-scale the variables to eliminate $|t|$'s and other coefficients.

$$a_i = \delta_i |t|^{\alpha_i} e^{-q_i}, \quad \zeta = \gamma t^\beta\,. \tag{4.20}$$

Then

$$\begin{aligned}
\alpha_1 &= -2\,, \qquad \alpha_2 = 2\beta - 4 \\
\alpha_m &= (m-1)(2\beta - 2) + \alpha_1 = -\frac{2(\beta-1)}{1+s}\,, \\
\beta &= 1 + \frac{1+s}{1 + (1+s)(m-1)} \\
\alpha_k &= -2\frac{(1+s)(m-k)+1}{1+(1+s)(m-1)} \\
\delta_1 &= 2\,, \qquad \delta_m = \delta_1 (\gamma\beta)^{2(m-1)} = (\gamma\beta)^{-\frac{1}{1+s}} \\
\gamma &= \beta^{-1}\left(\frac{1}{2}\right)^{\frac{1+s}{1+2(m-1)(1+s)}}\,.
\end{aligned} \tag{4.21}$$

Then

$$\begin{aligned}
\partial\bar{\partial} q_1 &= e^{q_1 - q_2} - e^{-(q_1+q_2)}\,, \\
\partial\bar{\partial} q_2 &= e^{q_2 - q_3} - e^{q_1 - q_2} - e^{-(q_1+q_2)}\,, \\
\partial\bar{\partial} q_i &= e^{q_i - q_{i+1}} - e^{-(q_{i-1} - q_i)}\,, \qquad i = 3, \ldots, m-1 \\
\partial\bar{\partial} q_m &= e^{(1+s)q_m} - e^{(q_{m-1} - q_m)}\,.
\end{aligned} \tag{4.22}$$

One can bring this equation to the Toda equations: for $s = 0$ (even n) these are B_m Toda and for $s = 1$ (odd n) they are $\tilde{B}_m = D^T(\mathrm{SO}(2m+1))$ Toda system. It is a curious fact that the Toda system appearing in the deformation of $\mathbb{C}^2/Z_n$ by xy is identical to that appearing in the deformation of D_n singularity ($W = \frac{1}{n-1}x^{n-1} + xy^2$) by x. Notice that the mirror geometric correspondence of D_n singularity has not been clear at all and is still not clear.

The charge matrix $Q = g\partial_\tau g^{-1} - 1$ with $\tau = \log\lambda$ can be calculated to be given by

$$Q = \begin{pmatrix} \frac{2-n}{n} + a_{11}\partial_\tau a_{11}^{-1} & 0 & \dots & 0 & 0 \\ 0 & \frac{4-n}{n} + a_{22}\partial_\tau a_{22}^{-1} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{n-4}{n} - a_{22}\partial_\tau a_{22}^{-1} & 0 \\ t^2 a_{11}\partial_\tau a_{11}^{-1} & 0 & \dots & 0 & \frac{n-2}{n} - a_{11}\partial_\tau a_{11}^{-1} \end{pmatrix}. \tag{4.23}$$

Notice that we do not need to perform the diagonalization to get the eigenvalues of Q. In terms of q_{ij} and $\lambda(=\zeta)$, and if we look at the $|\lambda| = z$ dependence only, the tachyon potential can be identified as

$$V = 2Q_{\max} = -z\partial_z q_{11}(z)\,. \tag{4.24}$$

So far, no mathematical literature on the solution to the equation is available. However, from the physical intuition that in the final stage of tachyon condensation there is no nontrivial chiral primaries with charge other than 0 and also from the experience from the low n cases, we expect that the potential monotonically decrease from the value $2/n - 1$ at $t = 0$ to 0 at $t \to \infty$.

5. Discussion

In this paper, we worked out equations describing the tachyon condensation in orbifolds $\mathbb{C}^1/\mathbb{Z}_n$ and $\mathbb{C}^2/\mathbb{Z}_n$ and the tachyon potential. The resulting equations can be identified as various Toda systems when we consider the condensation by the lowest tachyon. We find that the potential is calculated as a solution to Toda equations and conjectured to be monotonically decreasing.

It is interesting to notice that the Toda system appearing as tt^* equation in the deformation of $\mathbb{C}^2/\mathbb{Z}_n$ by xy is identical to that of D_n singularity by x. It is not clear whether this signalize the possibility to identify the geometry corresponding to D_n singularity as $\mathbb{C}^2/\mathbb{Z}_n$. It is tantalizing problem to answer to the question what is the geometry corresponding to the general singularities along the line of the idea that A_{N-1} singularity corresponds to the $\mathbb{C}/\mathbb{Z}_N$.

We should point out that we considered the string theories without GSO projection only when we discuss the xy deformation of $\mathbb{C}^2/\mathbb{Z}_n$ and x^2 deformation of $\mathbb{C}^1/\mathbb{Z}_n$. The reason is simply that if we impose the type-II projection rule, the above operators are projected out and remaining spectrum gives us complicated reality conditions so that the resulting tt^* equations are technically beyond our reach so far. Related problem arises when the daughter theory is also an orbifold $\mathbb{C}^1/\mathbb{Z}_n$ background.

One of the prime difficulty in the solution comes from the constraints coming from the reality condition. For the lowest deformations of $\mathbb{C}/Z_n$ and $\mathbb{C}^2/\mathbb{Z}_n$, the reality conditions can be easily solved as we have seen in the main sections. However, for $\mathbb{C}^3/\mathbb{Z}_n$ with lowest perturbation xyz or for $\mathbb{C}/\mathbb{Z}_n$ case with non-trivial deformation like x^3, the solutions are non-trivial. If we adopt an special ansatz, it the resulting solutions corresponds to the wrong boundary conditions as we described at the end of the appendices.

Of course most immediate question is whether we can write down analytic solution of the toda systems we found with appropriate boundary conditions. So far no mathematical literature is available to our knowledge. We wish to report on these issues in later publications.

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A. $\mathbb{C}^3/\mathbb{Z}_n$

In this appendix we present a somewhat incomplete result on $\mathbb{C}^3/\mathbb{Z}_n$. We spent long time without success in getting a simple system with solutions satisfying proper boundary conditions. We point out what are the difficulty through an example $n = 5$ case. We also give a comparison with similar system $\mathbb{C}/\mathbb{Z}_5 \to \mathbb{C}/\mathbb{Z}_3$.

The superpotential is

$$W = \frac{x^n}{n} + \frac{y^n}{n} + \frac{z^n}{n} - txyz\,. \tag{A.1}$$

We work in the basis

$$\{xyz, (xyz)^2, \ldots, (xyz)^{n-1}\}\,. \tag{A.2}$$

Under the transformation

$$x \to \omega^a x\,, \qquad y \to \omega^b y\,, \qquad z \to \omega^c z\,, \tag{A.3}$$

with well chosen a, b, and c, the superpotential becomes invariant. The choice is

$$b = a\,, \qquad c = a(n-2)\,, \qquad \text{with } \omega^{an(n-3)} = 1\,. \tag{A.4}$$

From this we can get the non-vanishing components of the topological metric η_{ij} and metric $g_{\bar{j}i}$. First for η_{ij},

$$\eta = \int \frac{(xyz)^I dx \wedge dy \wedge dz}{(x^{n-1} - tyz)(y^{n-1} - txz)(z^{n-1} - txy)}\,. \tag{A.5}$$

In order for this to be invariant under eq. (A.4),

$$an(I-1) = 0 \bmod an(n-3) \tag{A.6}$$

which means some components of η are non-zero. One can find and calculate the non-vanishing values of η:

$$0: \quad \text{when } I = 1 \qquad \text{and} \qquad n-1:, \quad t^3: \quad \text{when } I = 2n-3\,. \tag{A.7}$$

For the metric

$$g_{\bar{j}i} = \langle (xyz)^{\bar{j}} | (xyz)^i \rangle \tag{A.8}$$

to be invaraint we obtain

$$an(i-j)=0\,,\mathrm{mod}an(n-3)\,. \tag{A.9}$$

Therefore we obtain the diagonal of the metric (we spell out them as a_{ii}) and the off-diagonal components (denoted by b_1 and b_2 and their conjugates).

One can find C_t:

$$(C_t)_1^2=(C_t)_2^3=\cdots=(C_t)_{n-3}^{n-2}=1\,,\qquad (C_t)_3^{n-1}=t^3\,. \tag{A.10}$$

The difficult part is the reality condition. Although we can solve it with the ansatz

$$b_1=\frac{a_{11}t^3}{2}\,,\qquad b_2=\frac{a_{22}t^3}{2}\,, \tag{A.11}$$

there is no guarantee that its consequence is consistent with the boundary conditions we have to impose. With all these we can write down the tt^* equation, but for $n\geq 6$ we get inconsistent equation like

$$a_{11}=0\ (n=6)\qquad \text{or}\qquad a_{11}a_{33}=0\ (n>6) \tag{A.12}$$

which violates regularity of the metric. The tt^* equation for general $\mathbb{C}^3/\mathbb{Z}_n$ is not simple. Main reason is the absence of the simple solution to the reality constraint. We give some detail for $n=5$.

A.1 tt^* euqation for $\mathbb{C}^3/\mathbb{Z}_5\to\mathbb{C}^3/\mathbb{Z}_3$

In this case the superpotential is written as

$$W=\frac{x^5}{5}+\frac{y^5}{5}+\frac{z^5}{5}-txyz\,. \tag{A.13}$$

The symmetry that the superpotential has is

$$x\to\omega^a x\qquad y\to\omega^a y\qquad z\to\omega^{3a}z\qquad \text{with } \omega^{10a}=1\,. \tag{A.14}$$

This restricts the topological metric and metric to be

$$\eta_{ij}=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&t^3\\1&0&t^3&0\end{pmatrix},\qquad g_{\bar{j}i}=\begin{pmatrix}a_{11}&0&\bar{b}_1&0\\0&a_{22}&0&\bar{b}_2\\b_1&0&a_{33}&0\\0&b_2&0&a_{44}\end{pmatrix}, \tag{A.15}$$

and the multiplication matrix for the perturbing chiral ring element xyz is

$$(C_t)_i^j=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&t^3&0\end{pmatrix}. \tag{A.16}$$

We take the basis as

$$\{(xyz),(xyz)^2,(xyz)^3,(xyz)^4\}\,. \tag{A.17}$$

We list here all tt^* equations for $\mathbb{C}^3/\mathbb{Z}_5$ case.

$$\partial_{\bar{t}}\left(\frac{a_3\partial_t a_1 - b_1\partial_t\bar{b}_1}{-a_1a_3+b_1\bar{b}_1}\right) = \frac{a_2a_3-b_1\bar{b}_2}{a_1a_3-b_1\bar{b}_1}\,, \qquad \partial_{\bar{t}}\left(\frac{b_1\partial_t a_3 - a_3\partial_t b_1}{a_1a_3-b_1\bar{b}_1}\right) = \frac{a_4b_1-a_3b_2}{-a_1a_3+b_1\bar{b}_1}\,,$$

$$\partial_{\bar{t}}\left(\frac{a_4\partial_t a_2 - b_2\partial_t\bar{b}_2}{-a_2a_4+b_2\bar{b}_2}\right) = $$
$$= \frac{-a_2^2a_3a_4 + a_1a_3^2(a_4-\bar{t}^3b_2) + a_2(a_4b_1+a_3b_2)\bar{b}_2 - b_1(a_3\bar{b}_1(a_4-\bar{t}^3b_2)+b_2\bar{b}_2^2)}{(a_1a_3-b_1\bar{b}_1)(a_2a_4-b_2\bar{b}_2)}\,,$$

$$\partial_{\bar{t}}\left(\frac{b_2\partial_t a_4 - a_4\partial_t b_2}{a_2a_4-b_2\bar{b}_2}\right) = $$
$$= \frac{-t^3a_3a_4b_1\bar{b}_1 + |t|^6a_3b_1\bar{b}_1b_2 + t^3a_1a_3^2(a_4-\bar{t}^2b_2) + a_2a_4(a_4b_1-a_3b_2) - a_4b_1b_2\bar{b}_2 + a_3b_2^2\bar{b}_2}{(a_1a_3-b_1\bar{b}_1)(a_2a_4-b_2\bar{b}_2)}\,,$$

$$\partial_{\bar{t}}\left(\frac{\bar{b}_1\partial_t a_1 - a_1\partial_t\bar{b}_1}{a_1a_3-b_1\bar{b}_1}\right) = $$
$$= \frac{a_1(-b_2\bar{b}_2^2 + a_3\bar{b}_1(-a_4+\bar{t}^3\bar{b}_2) + a_2(-|t|^6a_3\bar{b}_1 + a_4\bar{b}_2)}{(a_1a_3-b_1\bar{b}_1)(a_2a_4-b_2\bar{b}_2)} +$$
$$+ \frac{\bar{b}_1(-a_2^2a_4 + b_1\bar{b}_1(a_4-\bar{t}^3b_2-t^3b_2) + a_2(|t|^6b_1\bar{b}_1+b_2\bar{b}_2))}{(a_1a_3-b_1\bar{b}_1)(a_2a_4-b_2\bar{b}_2)}\,,$$

$$\partial_{\bar{t}}\left(\frac{a_1\partial_t a_3 - \bar{b}_1\partial_t b_1}{-a_1a_3+b_1\bar{b}_1}\right) = $$
$$= \frac{\bar{b}_1(a_2(|t|^6a_3b_1 - a_4b_2) + b_2^2\bar{b}_2 + a_3b_1(a_4-\bar{t}^3b_2-t^3\bar{b}_2))}{(a_1a_3-b_1\bar{b}_1)(a_2a_4-b_2\bar{b}_2)} +$$
$$+ \frac{a_1(a_2(-|t|^6a_3^2+a_4^2) - a_4b_2\bar{b}_2 + a_3^2(-a_4+\bar{t}^3b_2+t^3\bar{b}_2))}{(a_1a_3-b_1\bar{b}_1)(a_2a_4-b_2\bar{b}_2)}\,,$$

$$\partial_{\bar{t}}\left(\frac{\bar{b}_2\partial_t a_2 - a_2\partial_t\bar{b}_2}{-a_2a_4+b_2\bar{b}_2}\right) = $$
$$= \frac{a_1(a_2(\bar{t}^3a_3^2 - a_4\bar{b}_2) + \bar{b}_2(-a_3^2+b_2\bar{b}_2)) + \bar{b}_1(a_2^2a_4 + a_3b_1\bar{b}_2 - a_2(\bar{t}^3a_3b_1+b_2\bar{b}_2))}{(a_1a_3-b_1\bar{b}_1)(a_2a_4-b2\bar{b}_2)}\,,$$

$$\partial_{\bar{t}}\left(\frac{a_2\partial_t a_4 - \bar{b}_2\partial_t b_2}{-a_2a_4+b_2\bar{b}_2}\right) = $$
$$= \frac{a_1(a_2(|t|^6a_3^2-a_4^2) - (t^3a_3^2+a_4b_2)\bar{b}_2 + \bar{b}_1(a_2(-|t|^6a_3b_1+a_4b_2) + (t^3a_3b_1-b_2^2)\bar{b}_2)}{(a_1a_3-b_1\bar{b}_1)(a_2a_4-b_2\bar{b}_2)}\,. \qquad (A.18)$$

These equations should be supplemented by the constraints coming from the reality condition:

$$\begin{aligned}
(\bar{t}^2a_1+\bar{b}_1)(-t^3a_2+b_2) + a_1(a_4-t^3\bar{b}_2) &= 1,\\
a_2(a_3-t^3\bar{b}_1) + (-t^3a_1+b_1)(-\bar{t}^3a_2+\bar{b}_2) &= 1,\\
a_2(a_3-\bar{t}^3b_1) + b_1\bar{b}_2 &= 1,\\
\bar{b}_1b_2 + a_1(a_4-\bar{t}^3b_2) &= 1,\\
a_2\bar{b}_1 + a_1(-\bar{t}^3a_2+b_2) &= 0,\\
(a_3-\bar{t}^3b_1)(-t^3a_2+b_2) + b_1(a_4-t^3\bar{b}_2) &= 0,\\
(a_3-t^3\bar{b}_1)b_2 + (-t^3a_1+b_1)(a_4-\bar{t}^3b_2) &= 0.
\end{aligned} \qquad (A.19)$$

We may wonder whether we can find some ansatz to make the tt^* equations simple.

We will try an obvious one and show how it does not work. Let's try an ansatz given by

$$b_1 = a_1 t^3/2\,, \qquad b_2 = a_2 t^3/2\,. \tag{A.20}$$

Together with the reality condition above ansatz gives

$$a_3 = \frac{1}{a_2} + \frac{|t|^6 a_1}{4}\,, \qquad a_4 = \frac{1}{a_1} + \frac{|t|^6 a_2}{4}\,, \qquad \text{with } a_1 a_4 = a_2 a_3\,. \tag{A.21}$$

From tt^* equation we get the relation

$$a_2 = -\frac{4}{(|t|^6 a_1)}\,. \tag{A.22}$$

Then finally we get

$$-\partial_{\bar{t}}\partial_t \log a_1 = -\frac{4}{(|t|^6 a_1^2)}\,. \tag{A.23}$$

Now let us define

$$\zeta = t^{-2} \qquad \text{and} \qquad a_1 = e^{\phi}\,, \tag{A.24}$$

the eq. (A.23) becomes Liouville equation

$$\partial_{\bar{\zeta}}\partial_\zeta \phi = e^{-2\phi}\,. \tag{A.25}$$

This has an analytic solution. However this turns out to have incorrect behavior.

Let us rescale

$$x = \lambda^{1/5}\tilde{x}\,, \qquad y = \lambda^{1/5}\tilde{y}\,, \qquad z = \lambda^{1/5}\tilde{z} \qquad \lambda = t^{2/5}\,. \tag{A.26}$$

Then we have the super potential is rescaled to give

$$\tilde{W} = \lambda\left(\frac{\tilde{x}^5}{5} + \frac{\tilde{y}^5}{5} + \frac{\tilde{z}^5}{5} - \tilde{x}\tilde{y}\tilde{z}\right). \tag{A.27}$$

The charge can be calculated by the relation

$$Q = \tilde{g}\partial_\tau \tilde{g}^{-1} - 3/2\,. \tag{A.28}$$

The correlation functions are rescaled to give

$$a_i = \langle(\bar{x}\bar{y}\bar{z})^i|(xyz)^i\rangle = |\lambda|^{6i/5}\langle(\bar{\tilde{x}}\bar{\tilde{y}}\bar{\tilde{z}})^i|(\tilde{x}\tilde{y}\tilde{z})^i\rangle := |\lambda|^{6i/5} b_i\,, \tag{A.29}$$

where b_i are components of $\tilde{g}$. Therefore the charge is

$$Q_{\min} = \frac{1}{2} b_1 |\lambda|\partial_{|\lambda|} b_1^{-1} - \frac{3}{2} = -\frac{9}{10} - \frac{1}{2}|\lambda|\frac{da_1}{d|\lambda|} = -\frac{9}{10} + \frac{5}{2}|\zeta|\frac{da_1}{d|\zeta|}\,, \tag{A.30}$$

where we have used $\lambda = t^{2/5}$ and $\zeta = t^{-2}$. So, the required boundary conditions are

$$|\zeta|\frac{da_1}{d|\zeta|} \to 00 \quad \text{if } |\zeta| \to \infty\ (UV)\,, \qquad |\zeta|\frac{da_1}{d|\zeta|} \to \frac{18}{50} \quad \text{if } |\zeta| \to 0\ (IR)\,. \tag{A.31}$$

The general solution for Liouville equation

$$-\partial_\zeta \partial_{\bar{\zeta}} \phi + e^{-2\phi} = \frac{1-a}{2} \delta^{(2)}(z)\,, \tag{A.32}$$

is [15]

$$e^{-2\phi} = \frac{\partial A(\zeta) \partial B(\bar{\zeta})}{(1 - A(\zeta)B(\bar{\zeta}))^2}\,. \tag{A.33}$$

There are three types of solutions according to the region of a: Elliptic (a real), Parabolic ($a \to 0$), Hyperbolic (a pure imaginary). However, none of such solutions can match the boundary conditions. Therefore the ansatz given in eq. (A.20) does not work.

A.2 tt^* euqation for $\mathbb{C}/\mathbb{Z}_5 \to \mathbb{C}/\mathbb{Z}_3$

Similar problem arise when we study $\mathbb{C}^1/\mathbb{Z}_5$ with non-trivial tachyon perturbation, namely,

$$W = \frac{x^5}{5} - \frac{tx^3}{3}\,. \tag{A.34}$$

Therefore the basis is

$$\{x, x^2, x^3, x^4\}\,. \tag{A.35}$$

The symmetry restricts the topological metric and metric which is given by

$$\eta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & t \\ 1 & 0 & t & 0 \end{pmatrix}, \qquad g = \begin{pmatrix} a_1 & 0 & \bar{b}_1 & 0 \\ 0 & a_2 & 0 & \bar{b}_2 \\ b_1 & 0 & a_3 & 0 \\ 0 & b_2 & 0 & a_4 \end{pmatrix}, \tag{A.36}$$

and C_t is given by

$$C_t = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & t^2 & 0 \end{pmatrix}. \tag{A.37}$$

We take ansatz to solve reality condition

$$b_1 = \frac{a_1 t}{2}\,, \qquad b_2 = \frac{a_2 t}{2}\,, \tag{A.38}$$

and the result is

$$a_3 = \frac{1}{a_2} + \frac{|t|^2 a_1}{4}\,, \qquad a_4 = \frac{1}{a_1} + \frac{|t|^2 a_2}{4}\,, \qquad \text{with } a_1 a_4 = a_2 a_3\,. \tag{A.39}$$

tt^* equation gives one more restriction to a_2

$$a_2 = |t| a_1\,. \tag{A.40}$$

Then we finally get the simplified tt^* equation

$$-\partial_{\bar{t}} \partial_t \log a_1 = \frac{1}{a_1^2} - \frac{|t|^6 a_1^2}{16}\,. \tag{A.41}$$

If we define $a_1 = 1/y$, then the equation becomes

$$\partial_{\bar{t}}\partial_t \log y = y^2 - \frac{|t|^6}{16y^2}\,. \tag{A.42}$$

By introducing change of variables

$$\zeta = \left(\frac{2}{5}\right) t^{5/2}\,, \qquad y = \sqrt{\frac{1}{2}}|t|^{3/2} e^{u/2}\,, \tag{A.43}$$

one can rewite this equation as sinh-Gordon

$$\partial_{\bar{\zeta}}\partial_\zeta u = \sinh u\,. \tag{A.44}$$

The solution to this equation is describing the solution whose associated charges decrease to zero, which is not what we want here. Hence we see that the ansatz (A.38) above is not a proper one. Then the equation is given by tt^* equation

$$\begin{aligned}
&\partial_{\bar{t}}\left(\frac{a_3\partial_t a_1 - b_1\partial_t \bar{b}_1}{-a_1a_3 + |b_1|^2}\right) = \frac{a_4(a_3 - \bar{t}b_1)}{a_1a_3 - |b_1|^2}\,,\\
&\partial_{\bar{t}}\left(\frac{b_1\partial_t a_3 - a_3\partial_t b_1}{a_1a_3 - |b_1|^2}\right) = \frac{ta_4(a_3 - \bar{t}b_1)}{a_1a_3 - |b_1|^2}\,,\\
&\partial_{\bar{t}}\left(\frac{a_4\partial_t a_2 - b_2\partial_t \bar{b}_2}{-a_2a_4 + |b_2|^2}\right) = \frac{|t|^2 a_3(a_4 - \bar{t}b_2)}{a_2a_4 - |b_2|^2}\,,\\
&\partial_{\bar{t}}\left(\frac{b_2\partial_t a_4 - a_4\partial_t b_2}{a_2a_4 - |b_2|^2}\right) = \frac{t|t|^2 a_3(a_4 - \bar{t}b_2)}{a_2a_4 - |b_2|^2}\,,\\
&\partial_{\bar{t}}\left(\frac{\bar{b}_1\partial_t a_1 - a_1\partial_t \bar{b}_1}{a_1a_3 - |b_1|^2}\right) =\\
&\quad = \frac{\bar{t}a_1(a_2(a_4^2 - |t|^2 t a_3\bar{b}_1) - a_4|b_2|^2 + ta_3\bar{b}_1(-a_4 + \bar{t}b_2 + t\bar{b}_2))}{(a_1a_3 - |b_1|^2)(a_2a_4 - |b_2|^2)} +\\
&\qquad + \frac{\bar{b}_1(a_2(-a_4^2 + |t|^4|b_1|^2) - |t|^2|b_1|^2(\bar{t}b_2 + t\bar{b}_2 + a_4(|t|^2|b_1|^2 + |b_2|^2))}{(a_1a_3 - |b_1|^2)(a_2a_4 - |b_2|^2)}\,,\\
&\partial_{\bar{t}}\left(\frac{a_1\partial_t a_3 - \bar{b}_1\partial_t b_1}{-a_1a_3 + |b_1|^2}\right) =\\
&\quad = \frac{t(\bar{t}a_1(a_2(-|t|^2a_3^2 + a_4^2) - a_4|b_2|^2 + a_3^2(-a_4 + \bar{t}b_2 + t\bar{b}_2)))}{(a_1a_3 - |b_1|^2)(a_2a_4 - |b_2|^2)} +\\
&\qquad + \frac{t(\bar{b}_1(a_2(-a_4^2 + |t|^2\bar{t}a_3b_1) + a_4|b_2|^2 - \bar{t}a_3b_1(-a_4 + \bar{t}b_2 + t\bar{b}_2)))}{(a_1a_3 - |b_1|^2)(a_2a_4 - |b_2|^2)}\,,\\
&\partial_{\bar{t}}\left(\frac{\bar{b}_2\partial_t a_2 - a_2\partial_t \bar{b}_2}{a_2a_4 - |b_2|^2}\right) =\\
&\quad = \frac{a_2(a_4(\bar{t}b_1 + t\bar{b}_1)\bar{b}_2 - a_3(|t|^2\bar{t}|b_1|^2 + a_4\bar{b}_2))}{(a_1a_3 - |b_1|^2)(a_2a_4 - |b_2|^2)} +\\
&\qquad + \frac{|t|^2a_1(a_2(\bar{t}a_3^2 - a_4\bar{b}_2) + \bar{b}_2(-a_3^2 + |b_2|^2)) + \bar{b}_2(-(\bar{t}b_1 + t\bar{b}_1)|b_2|^2 + a_3(|t|^2|b_1|^2 + |b_2|^2))}{(a_1a_3 - |b_1|^2)(a_2a_4 - |b_2|^2)}\,,\\
&\partial_{\bar{t}}\left(\frac{a_2\partial_t a_4 - \bar{b}_2\partial_t b_2}{-a_2a_4 + |b_2|^2}\right) =\\
&\quad = \frac{a_2(a_4^2(\bar{t}b_1 + t\bar{b}_1) - a_3(a_4^2 + |t|^4|b_1|^2))}{(a_1a_3 - |b_1|^2)(a_2a_4 - |b_2|^2)} +\\
&\qquad + \frac{(-a_4(\bar{t}b_1 + t\bar{b}_1)b_2 + a_3(|t|^2t|b_1|^2 + a_4b_2))\bar{b}_2 + |t|^2a_1(a_2(|t|^2a_3^2 - a_4^2) + (-ta_3^2 + a_4b_2)\bar{b}_2)}{(a_1a_3 - |b_1|^2)(a_2a_4 - |b_2|^2)}
\end{aligned} \tag{A.45}$$

substantiated by the constraints coming from reality condition

$$\begin{aligned}
(-\bar{t}a_1+\bar{b}_1)(-ta_2+b_2)+a_1(a_4-t\bar{b}_2) &= 1\,,\\
a_2(a_3-t\bar{b}_1)+(-ta_1+b_1)(-\bar{t}a_2+\bar{b}_2) &= 1\,,\\
a_2(a_3-\bar{t}b_1)+b_1\bar{b}_2 &= 1\,,\\
\bar{b}_1b_2+a_1(a_4-\bar{t}b_2) &= 1\,,\\
a_2\bar{b}_1+a_1(-\bar{t}a_2+\bar{b}_2) &= 0\,,\\
(a_3-\bar{t}b_1)(-ta_2+b_2)+(b_1(a_4-t\bar{b}_2)) &= 0\,,\\
(a_3-t\bar{b}_1)b_2+(-ta_1+b_1)(a_4-\bar{t}b_2) &= 0\,.
\end{aligned} \tag{A.46}$$

The examination of this constrained and coupled non-linear system is out of the scope of this paper, although it should be a integrable one from the zero curvature structure of tt^* equation.

B. A sample residue calculation

Here we give a sample calculation for the residue.

$$K=\langle (xy)^{n-1}|(xy)^{n-1}\rangle=\int dx\wedge dy\frac{(xy)^{2(n-2)}}{(x^{n-1}-ty)(y^{n-1}-tx)}\,.$$

We evaluate the residue iteratively, that is, the vanishing of the denominator gives $y_{0i}=(tx)^{1/n-1}\omega^i$ with $\omega^{n-1}=1$ and $y_1=x^{n-1}/t$. Then the residue integral of the y becomes

$$\begin{aligned}
K &= -\int dx\sum_{i=0}^{n-1}\frac{x^{2(n-2)}y_{0i}^{2(n-2)}}{(n-1)y_{0i}^{n-2}(x^{n-1}-ty_{0i})}+\int dx\frac{x^{2(n-2)}{y_1}^{2(n-2)}}{t(y_1^{n-1}-tx)}\\
&= -\sum_{i=0}^{n-2}\frac{1}{n-1}\int dx\frac{x^{2(n-2)}(tx)^{(n-2)/(n-1)}\omega^{i(n-2)}}{x^{n-1}-t(tx)^{1/(n-1)}\omega^i}+\frac{1}{t}\int dx\frac{(x^{n-1}/t)^{2(n-2)}x^{2(n-2)}}{(x^{n-1}/t)^{n-1}-tx}\,.
\end{aligned}$$

The first integral is zero since it can be brought to the polynomial denominator by change of variable $x=u^{n-1}$. But then the u contour does not close the poles when x wrap a circle. The second integral becomes

$$\begin{aligned}
(1/t)\int dx\frac{t^{-2(n-2)}x^{2(n-2)(n-1+1)-1}}{t^{1-n}x^{n(n-2)}-t} &= (1/t)t^{-2(n-2)}\sum_{i=0}^{n(n-2)-1}\frac{x^{2(n-2)n-1}}{t^{1-n}n(n-2)x^{n(n-2)-1}}\Big|_{x=x_i}\\
&= t^2\,,
\end{aligned}$$

where x_i is a solution of $x^{n(n-2)}=t^n$.

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